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Non(anti)commutative superspace, UV/IR mixing & open Wilson lines

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Abstract: We study quantum aspects of field theories defined on $\mathcal{N} = 1/2$ superspace, where both bosonic and fermionic coordinates are made non(anti)commutative. We compute the one-loop effective superpotential, and we find that planar and nonplanar contributions exhibit markedly different behavior. Planar diagrams yield an effective superpotential proportional to $N_c(\Phi \log \Phi)_\star$. For nonplanar diagrams, we show that ultraviolet-infrared mixing takes place and explain why some nonplanar diagrams are ultraviolet-divergent when bosonic noncommutativity is turned off. Each nonplanar diagram is not expressible as a star product of background fields, but, once resummed appropriately, they are expressed as a star product involving open Wilson lines in superspace. The open Wilson lines are responsible for ultraviolet-infrared mixing. We comment on an intriguing relation of our result to the Dijkgraaf-Vafa correspondence between gauge theories and matrix models.

Keywords: Supersymmetric Effective Theories, Non-Commutative Geometry.

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1. Introduction

D-brane dynamics in the background of a closed string p-form gauge field has been a source of surprises. When a flat potential of the Kalb-Ramond field B_{NS} is turned on, it was discovered that open string dynamics perceives noncommutative spacetime [1], whose coordinates obey the Heisenberg algebra,

$$[y^m, y^n] = i\Theta^{mn} \,. \tag{1.1}$$

Moreover, in the Seiberg-Witten scaling limit [2], excitations of all closed string modes and massive open string modes are decoupled from the low-energy dynamics on the D-brane. As a result, there emerges on the D-brane worldvolume a new kind of theories, referred to as noncommutative field theories. These theories are now known to exhibit many surprising features, such as ultraviolet(UV)-infrared(IR) mixing [3], nonlocal open Wilson lines as physical observables [4] and as a sort of master fields. With these features, noncommutative field theories depart from the ordinary field theories but behave more like fundamental string theories.

Given that noncommutative space emerged from (super) string theories, can non(anti)-commutative superspace emerge from superstring theories? Recently, in the context of the Dijkgraaf-Vafa correspondence [5] relating four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories and zero-dimensional matrix models, it was suggested that non(anti)commutative superspace indeed emerges on D-brane worldvolume if one turns on self-dual graviphoton field strength in four dimensions [6, 7, 8] or, more generally, Ramond-Ramond 2-form field strength in ten dimensions [9]. The Grassmann-odd coordinates $\theta^\alpha (\alpha = 1, 2)$ are now non(anti)commuting [10, 11, 7] and obey a Clifford algebra:

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} \,. \tag{1.2}$$

The development prompts the study of quantum field theories defined on non(anti)-commutative superspace. In [7], it was pointed out that these theories constitute non(anti)-commutative supersymmetric field theories, in which the ordinary product is replaced by a $\star$-product:

$$\star = \exp\left(-\frac{i}{2}\Theta^{mn} \frac{\overleftarrow{\partial}}{\partial y^m} \frac{\overrightarrow{\partial}}{\partial y^n} - \frac{1}{2} C^{\alpha\beta} \frac{\overleftarrow{\partial}}{\partial \theta^\alpha} \frac{\overrightarrow{\partial}}{\partial \theta^\beta} \right) . \tag{1.3}$$

It was also pointed out that, with non(anti)commutativity turned on, the $\mathcal{N} = 1$ supersymmetry in four-dimensional euclidean space is broken to $\mathcal{N} = 1/2$ supersymmetry, but nevertheless preserves the antichiral ring structure. In previous work [12], we studied quantum aspects of non(anti)commutative field theories (see also [13]), and established the $\mathcal{N} = 1/2$ (non)renormalization theorem. According to the theorem, the antiholomorphic superpotential ($\overline{\text{F}}$-term) is not renormalized, while the holomorphic superpotential (F-term) is subject to renormalization and is combined to the Kähler potential (D-term). Nevertheless, the energy density of supersymmetric vacua still vanishes to all orders in perturbation theory.

Being subject to renormalization, the superpotential (F-term) is radiatively corrected by terms like $\Phi Q^2 \Phi$. As it stands, such a term is not expressible in terms of the $\star$-product eq. (1.3), as the generic $\star$-product among superfields produces only *even* powers of Q^2. One might regard it (as in [13]) as indicating that the $\star$-product eq. (1.3) one started with is no longer valid quantum-mechanically, and hence needs to be modified to some sort of generalized $\star$-product in the effective superpotential. Quite to the contrary, in this work, we show that the $\star$-product eq. (1.3) performs just as well at the quantum level; rather, the operators need to be reorganized to a set of nonlocal objects called open Wilson lines. Specifically, in the effective superpotential, nonplanar diagrams produce local operators of increasing powers of the Φ field and Q^2 (as well as $\Box$'s) acting on them, of which the aforementioned $\Phi Q^2 \Phi$ is one of the lowest dimension operators. Each local operator is indeed not expressible in terms of the $\star$-product eq. (1.3), but a suitable resummation of a set of local operators is. We thus resolve the conundrum by demonstrating that, by resumming individual terms, the effective superpotential may be organized as a generating functional of the open Wilson lines. Importantly, in defining the open Wilson line, no modification to the $\star$-product eq. (1.3) is necessary.

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Essentially the same conundrum arose in the context of noncommutative field theory [14], where, initially, a class of generalized $\star$-products was considered inevitable for expressing quantum effects. It was then found in [15] that the quantum effect is not to modify the definition of the $\star$-product, but to reorganize the local operators into open Wilson lines. It is now well understood that the open Wilson line, and the intuitive picture of it as an analog of electric dipole in a magnetic field, are the fundamental reasons that underlie the UV-IR mixing in ordinary noncommutative field theories [16].

As said, the crux of the present work is to demonstrate that quantum effect in non(anti)commutative field theories is not to modify the $\star$-product, but to reorganize local operators into open Wilson lines. For simplicity, in this work, we shall consider the $\mathcal{N} = 1/2$ Wess-Zumino model whose holomorphic action is given by

$$S = \int \mathrm{d}^4y \mathrm{d}^2\theta \left[\frac{1}{2}\Phi \star \left(\left(\frac{1}{4\lambda}Q^2\right) - \frac{\Box}{\overline{m}} + m\right) \star \Phi + \frac{g}{3}\Phi \star \Phi \star \Phi + \cdots\right],$$

in a suitable kinematical limit. We find, significantly, that the open Wilson lines extend over the superspace. So, for example, the open Wilson line carrying supermomentum (k, κ) takes the form[1]

$$\mathcal{W}_{k,\kappa}[\Phi] = \mathcal{P}_\sigma \int \mathrm{d}^4y \mathrm{d}^2\theta \, \exp_\star\left(-\int_0^1 \sqrt{h}\, \mathrm{d}\sigma \, \Phi(y + \Theta k\, \sigma,\, \theta - C\kappa\, \sigma)\right) \star e^{-iky - \kappa\theta}. \tag{1.4}$$

Notice that the open Wilson line extends in (y, θ) superspace over the interval $(\Theta k,\, C\kappa)$, and hence encodes the $\mathcal{N} = 1/2$ version of the aforementioned *dipole relations*:

$$\Delta y^m = \Theta^{mn} k_n \quad \text{and} \quad \Delta\theta^\alpha = C^{\alpha\beta}\kappa_\beta \,. \tag{1.5}$$

Noting that (k, κ) refers to the total momentum of the open Wilson line, it is evident from these relations that there emerges a UV-IR relation in *superspace*.

This paper is organized as follows. In section 2, we formulate the holomorphic Wess-Zumino model eq. (2.1) by integrating out the $\overline{\Phi}$ field. We set up the one-loop background field method and derive Feynman rules thereof. In section 3, we compute the effective superpotential using the background field method. In section 4, we explain our result and discuss in detail various limits of interest, where the non(anti)commutativity is turned off. In section 5, we resum planar diagrams and show that they give rise to a contribution proportional to $N_c(\Phi \log \Phi)_\star$. Appendix A contains notation, conventions, and some details of Fourier transformation in $\mathcal{N} = 1/2$ superspace. Appendix B contains the detailed derivation of various results sketched in section 3.

2. $\mathcal{N} = 1/2$ SUSY Wess-Zumino model

We begin with recollecting that the non(anti)commutative Wess-Zumino model is defined as the ordinary Wess-Zumino model in deformed superspace, whose coordinates obey Heisen-

[1] In this expression, $h = (g/2)^2(\overline{m}/m)|\Theta k|^2$ is a kinematic factor defining a sort of the metric in the σ-parameter space. See below.

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berg/Clifford algebras, eqs. (1.1), (1.2):

$$S = \int \mathrm{d}^4y \Big[\int \mathrm{d}^2\theta \mathrm{d}^2\overline{\theta}\, \Phi \star \overline{\Phi} + \int \mathrm{d}^2\theta \Big(\frac{m}{2} \Phi \star \Phi + \frac{g}{3} \Phi \star \Phi \star \Phi \Big) + \\ + \int \mathrm{d}^2\overline{\theta} \Big(\frac{\overline{m}}{2} \overline{\Phi} \star \overline{\Phi} + \frac{\overline{g}}{3} \overline{\Phi} \star \overline{\Phi} \star \overline{\Phi} \Big) \Big] ,$$

where the $\star$-product is as defined in eq. (1.3). We have proven in [12] that the antiholomorphic superpotential $\overline{W}_\star(\overline{\Phi})$ is not renormalized to all orders in perturbation theory. We will therefore study the effective holomorphic superpotential $W_\star^{\mathrm{eff}}(\Phi)$ that is generated by quantum effects. The situation is similar to the diagrammatic derivation [17] of the Dijkgraaf-Vafa correspondence between gauge theory and matrix model: both cases concern computation of quantities governed and generated by the holomorphic superpotential. As in [17], to simplify the computation, we will set $\overline{g}$ (as well as couplings of all higher monomials) to zero. We shall comment below on the consequences when these antiholomorphic nonlinear couplings are non-vanishing. Integrating out the $\overline{\Phi}$ field amounts to doing a gaussian integral (the manipulation is standard and elementary, as the D-term is just a nonchiral coupling to the 'external source' Φ), and results in a term proportional to $\Phi\Box\Phi$. We will also add the $\mathcal{N} = 1/2$ kinetic multiplet term, $\Phi Q^2 \Phi$. In [12], we demonstrated that this term is generated at one loop. In fact, these two are the most general kinetic terms preserving $\mathcal{N} = 1/2$ supersymmetry. Putting it all together, the action of holomorphic Wess-Zumino model may be written as

$$S = \int \mathrm{d}^4y \mathrm{d}^2\theta \left[\frac{1}{2} \Phi(y,\theta) \left(\frac{1}{4\lambda} Q^2 - \frac{1}{\overline{m}} \Box + m \right) \Phi(y,\theta) + \frac{g}{3} \Phi(y,\theta) \star \Phi(y,\theta) \star \Phi(y,\theta) \right] . \tag{2.1}$$

Here, we have introduced a dimensionless coupling constant λ to the kinetic multiplet term, and we will treat it as a variable parameter along with the others, $m, \overline{m}$ and g.

In computing N-point one-particle-irreducible Green functions, as in noncommutative field theories, we find it convenient to work in momentum superspace and write the $\star$-product as a momentum-dependent phase factor. The Φ-field propagator is given by

$$\langle \widetilde{\Phi}(k,\kappa) \widetilde{\Phi}(k',\kappa') \rangle = \Delta(k,\kappa) (2\pi)^4 \delta^4(k+k') \left(\frac{i}{2} \right)^2 \delta^2(\kappa+\kappa')$$

where

$$\Delta(k,\kappa) = \frac{1}{\frac{1}{\overline{m}} k^2 + \frac{1}{4\lambda} \kappa^2 + m} .$$

Denote[2] superspace coordinates as $Y = (y,\theta)$ and corresponding momenta as $K = (ik,\kappa)$. Then the action eq. (2.1) can be written in momentum space as

$$S = \frac{1}{2} \int \prod_{i=1,2} \mathrm{d}^6 K_i \widetilde{\Phi}(K_i)\, \Delta^{-1}(K_1)\, \delta^6(K_1+K_2) + \\ + \frac{g}{3} \int \prod_{i=1,2,3} \mathrm{d}^6 K_i \widetilde{\Phi}(K_i)\; e^{\sum_{i<j} \frac{i}{2} K_i \wedge K_j}\, \delta^6(K_1+K_2+K_3) . \tag{2.2}$$

[2] Our notation and conventions are explained in detail in appendix A.

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Two remarks are in order.

- As in [17], we have set $\overline{g}$ to zero. This is only a mild convenience and not a severe restriction. Indeed, suppose we leave $\overline{g}$ finite and integrate out $\overline{\Phi}$. We then get an infinite series of additional terms containing three or higher powers of Φ. They are generated by connected tree diagrams of $\overline{\Phi}$'s, with propagator $1/\overline{m}$ and interaction vertex $\overline{g}$, upon converting external $\overline{\Phi}$ fields to Φ fields via the D-term $[\Phi\overline{\Phi}]_{\theta\theta\overline{\theta}\overline{\theta}}$. In doing so, we find that all these terms involve a laplacian $\Box_y$ and multiples of Q^2's acting on Φ fields. Thus, with $\overline{g}$ nonzero, the holomorphic superpotential $W_\star(\Phi)$ is not modified, and only interaction terms involving derivatives are generated. As they will either produce extra contact terms or higher-derivative corrections to the open Wilson line, and do not entail any new physics to the results we derive, we do not consider these terms further.

- It is straightforward to extend the superfield Φ to a matrix-valued one and couple it to an external gauge field. In particular, if Φ belongs to the adjoint representation of the gauge group $\mathrm{U}(N)$ (or the bifundamental representation of the quiver gauge group), the holomorphic Wess-Zumino model is equivalent via Eguchi-Kawai reduction to a super-matrix model.[3] In this case, the superspace integrals in the action and open Wilson lines include a trace over the color indices of the Φ-field.

Computation of the effective action at one loop is best facilitated by the background field expansion. Split the superfield Φ into classical background and quantum fluctuation:

$$\Phi(Y) = \Phi_0(Y) + \varphi(Y)\,.$$

With fully symmetrized labelling of the momentum, the interaction term is given by (keeping only the quadratic term of ϕ)

$$g\int \mathrm{d}^6K_1\cdots\mathrm{d}^6K_3\delta^6(K_1+K_2+K_3)\,[U+T]\,\varphi(K_1)\Phi_0(K_2)\varphi(K_3)\,, \tag{2.3}$$

where U, T are phase factors:

$$\begin{aligned}\mathrm{U}(K_1,K_2) &= \exp\left(-\frac{i}{2}k_1\wedge k_2-\frac{1}{2}\kappa_1\wedge\kappa_2\right)\\ T(K_1,K_2) &= \exp\left(+\frac{i}{2}k_1\wedge k_2+\frac{1}{2}\kappa_1\wedge\kappa_2\right).\end{aligned} \tag{2.4}$$

The two phase factors differ by a relative sign in the exponent, and we will call them the 'untwisted' and the 'twisted' vertex, respectively. They are hermitean conjugates. Notice that the relative signs of bosonic and fermionic momentum phase factors in U, T are *correlated*, so they are untwisted or twisted simultaneously. This property will be the crucial ingredient in our computations.

[3] This is in spirit related to the observation made in [11].

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3. The one-loop effective superpotential

The action eq. (2.1), or equivalently, eq. (2.2), is our starting point for the one-loop computation of the effective superpotential in the Φ-field background. As mentioned, we will utilize the background field expansion around $\Phi = \Phi_0(Y)$. The one-loop computation proceeds as follows. Integrating out the fluctuation field φ yields schematically

$$\mathrm{Tr} \int \mathrm{d}^6 K \log \left[\Delta^{-1}(K) + \star g \widetilde{\Phi}_0 \star \right] .$$

Expanding it in powers of the background field Φ_0 gives rise to insertion of the interaction vertices to the one-loop vacuum diagram. Denote the super-momentum circulating the loop as $P = (ip, \pi)$. The N-point function follows from the expansion as

$$\begin{aligned} G_N \,=\, & \left(-\frac{1}{N!} \right) (-g)^N \int \mathrm{d}^6 K_1 \cdots \mathrm{d}^6 K_N \, \widetilde{\Phi}_0(K_1) \cdots \widetilde{\Phi}_0(K_N) \, \delta^6(K_1 + \cdots + K_N) \times \\ & \times \int \mathrm{d}^6 P \prod_{i=1}^{N} \Delta \left(p + \sum_{j=1}^{i} k_j, \pi + \sum_{j=1}^{i} \kappa_j \right) \times \\ & \times \prod_{i=1}^{N} \exp \left[\frac{\epsilon_i}{2} \left(\kappa_i \wedge (\pi + \sum_{j=1}^{i} \kappa_j) + i k_i \wedge (p + \sum_{j=1}^{i} k_j) \right) \right] , \end{aligned} \tag{3.1}$$

where $\epsilon = \pm 1$ is the relative sign for the un/twisted vertex. The first line contains vertex symmetry factors, coupling parameter, background fields, and overall super-momentum conservation. The second line is the loop momentum integration times N propagators connecting adjacent background field vertex pairs, and the third line is the phase factor U or T originating from the $\star$-product.

We shall use the double-line notation for the propagators and the interaction vertices to distinguish the untwisted/twisted vertices. See figure 1*a*. In this notation, the untwisted/twisted vertex injects momentum to the inner/outer boundary. Evidently, there are 2^N combinatorial possibilities, each vertex being either untwisted or twisted. For each diagram contributing to the N-point function, define the net momentum flow to the inner boundary as

$$k_0 \equiv \sum_{i \in \{U\}} k_i \,, \qquad \kappa_0 \equiv \sum_{i \in \{U\}} \kappa_i \,,$$

where $i \in \{U\}$ denotes those vertices i chosen to be untwisted. Overall momentum conservation implies that they are the same as *minus* the net momentum flow to the outer boundary:

$$k_0 = - \sum_{i \in \{T\}} k_i \,, \qquad \kappa_0 = - \sum_{i \in \{T\}} \kappa_i \,.$$

Our next task is to perform the loop momentum integrals and rearrange the amplitude in the cross-channel, as depicted in figure 1*b*. Since channel duality is the feature observed in string theory, as in the noncommutative field theory computations, we express the

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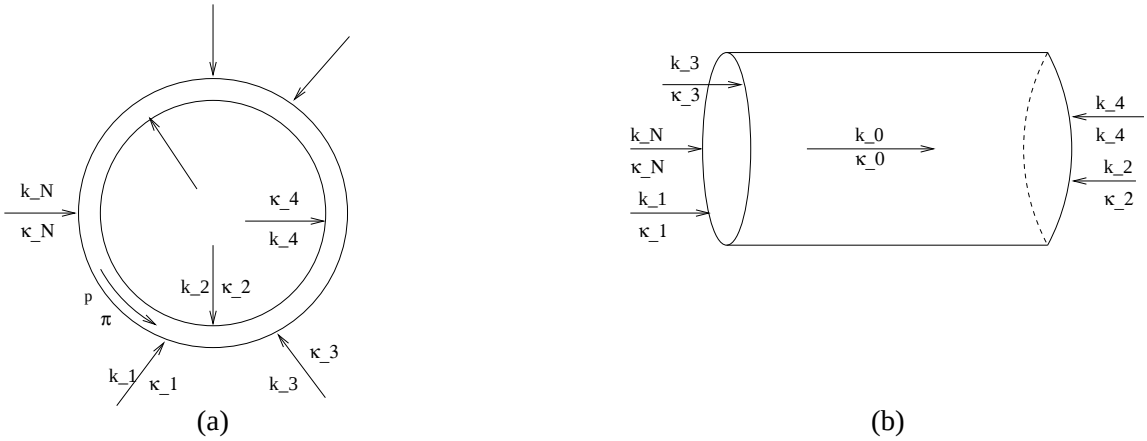

Figure 1: *a* The one loop N-point function. Double-line notation is featured in the loop to distinguish twisted and untwisted vertices. *b* Stretching the cylinder illustrates our eventual understanding in terms of open Wilson lines for untwisted and twisted vertices separately which hints at underlying open-closed duality.

Feynman diagram in terms of Schwinger time variables by expressing the i-th propagator in eq. (3.1) as

$$\int \mathrm{d}s_i \exp\left[-s_i\overline{m}\Delta^{-1}\left(p+\sum_{j=1}^{i}k_j, \pi+\sum_{j=1}^{i}\kappa_j\right)\right].$$

The Schwinger parameter s_i can be thought of as the length of the i-th arc, or the distance between the $(i-1)$-th and i-th vertices in the Feynman graph. For the N-point amplitude, we have N propagators and hence N Schwinger parameters. These Schwinger parameters span the moduli space of N marked points on the one-loop vacuum diagram. Equivalently, they can be viewed as parametrizing the moduli space of the perimeter of the one-loop vacuum diagram and $(N-1)$ relative marked points on a circle of unit radius. The perimeter is measured by $s \equiv (s_1 + \cdots + s_N)$.

With Schwinger parameters introduced, integration over the loop momentum P is reduced to a gaussian integral. In doing so, as seen from the product of N propagators eq. (3), the coefficient of P^2 is s. So, factoring out s from the complete-square for P, the residual terms are proportional to $\sigma_i \equiv (s_i/s)$. These σ_i's are precisely the moduli of relative marked points on a unit circle. Integration over P now yields $s^{-2} \times s$ from bosonic and fermionic integrals.

The residual terms in the exponent depends on the external momenta K_i in a complicated and unilluminating way. We are however interested in the effective superpotential — terms that do not depend on derivatives other than those in the $\star$-product. So we will isolate these terms by taking the limit of large non(anti)commutativity and small external momenta:

$$k_i, \kappa_i \to \mathcal{O}(\epsilon) \quad \text{and} \quad \Theta^{\mu\nu},\, C^{\alpha\beta} \to \mathcal{O}(\epsilon^{-2})\,. \tag{3.2}$$

In fact, we shall see later that this limit is quite harmless, because we will still reproduce the known results in the limit that either Θ or C is taken back to zero (along with λ, since

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this feature is new in our analysis). The result for the N-point Green function is quite simply

$$\begin{aligned} G_N \;=\; & \frac{1}{4\pi^2}\frac{\overline{m}}{4\lambda}\frac{(-g\overline{m})^N}{N!}\int\prod_{i=1}^{N}\mathrm{d}^6K_i\,\widetilde{\Phi}_0(K_i)\,\delta^6(K_1+\cdots+K_N)\,\times \\ & \times\int_0^\infty \mathrm{d}s\;s^{N-2}\exp\left[-s\,m\overline{m}-\frac{1}{4s}\left((\Theta k_0)^2-\frac{4\lambda}{\overline{m}}(C\kappa_0)^2\right)\right] \\ & \times\int_0^{\sigma_1}\mathrm{d}\sigma_2\cdots\int_0^{\sigma_{N-1}}\mathrm{d}\sigma_N\left(e^{+i(\Theta k_0)\sum_k\sigma_k k_k+\frac{i}{2}\sum_{k\le j}\frac{\epsilon_j+\epsilon_k}{2}k_j\wedge k_k}\right)\times \\ & \times\left(e^{-(C\kappa_0)\sum_k\sigma_k\kappa_k+\frac{1}{2}\sum_{k\le j}\frac{\epsilon_j+\epsilon_k}{2}\kappa_j\wedge\kappa_k}\right). \end{aligned} \tag{3.3}$$

We identify two separate pieces: the vacuum moduli contribution in the second line, and the phase factors from non(anti)commutativity in the third line. Appendix B contains some details of the derivation of eq. (3.3).

In eq. (3.3) above, the integration over the relative moduli is specified for a particular ordering of given n untwisted and $(N-n)$ twisted background field insertions. If we sum over all possible orderings, the effect is just to extend every σ_i integral over $[0,1]$. Moreover, and remarkably, the phase factor in eq. (3.3) is factorized into those for untwisted insertions and those for twisted ones. This is very much like open string annulus amplitudes, and eventually enables us to re-express the amplitudes in the cross-channel as in figure 1*b*.

The effective superpotential is obtained by summing all N-point contributions. Taking into account the combinatoric factor $\binom{N}{n}$ for the N-point Green function with n untwisted vertices, the result is

$$W_{\text{eff}}[\Phi_0]=\frac{m\overline{m}}{4\pi^2}\int_0^\infty\frac{\mathrm{d}\tau}{\tau^2}\int\mathrm{d}^6K_0\,\mathcal{W}_{K_0,\tau}[\Phi]\mathcal{K}(K_0;\tau)\mathcal{W}_{-K_0,\tau}[\Phi]\,, \tag{3.4}$$

where we have introduced the dimensionless overall modulus $\tau=m\overline{m}s$; the 'scalar' open Wilson line

$$\mathcal{W}_{K_0,\tau}[\Phi]=\mathrm{Tr}\int\mathrm{d}^6Y\mathcal{P}_\sigma\exp_\star\left(-\frac{g\tau}{m}\int_0^1\mathrm{d}\sigma\;\Phi(y+(\Theta k_0)\sigma,\;\theta-(C\kappa_0)\sigma)\right)\star e^{-ik_0\cdot y-\kappa_0\cdot\theta}, \tag{3.5}$$

where 'Tr' refers to trace over color indices in case the scalar superfield Φ is taken to be U(N) matrix-valued; and the 'cross-channel propagator'

$$\mathcal{K}(K_0;\tau)=\left(\frac{\overline{m}}{4\lambda}+\frac{L}{\tau}\kappa_0^2\right)\exp\left(-\tau-\frac{M^2}{\tau}\right), \tag{3.6}$$

where L, M are dimensionless combinations of the non(anti)commutativity parameters:

$$L\equiv\frac{1}{4}m\overline{m}\det(C)\,,\qquad M\equiv\frac{1}{2}\sqrt{m\overline{m}}|\Theta k_0|\,.$$

It is worth noting that the open Wilson line defined by eq. (3.5) depends on the overall modulus τ, and the final result is the weighted sum of all open Wilson lines in τ-space. Roughly speaking, the parameter σ is the affine parameter around each index-loop, and the parameter τ sets the size of the index-loop in superspace.

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The manifestly channel-dual expression eq. (3.4) of the one-loop effective superpotential is the central result. As in the noncommutative field theories, by identifying the open Wilson line as a closed string field, the effective action takes strikingly the same form as the quadratic term in the closed string field theory. In fact, adopting the result [19], we expect that the ℓ-loop contribution to the effective superpotential is expressible as an interaction involving ℓ closed string fields.

We will base our discussion on eq. (3.4) and explore further aspects of the channel duality.

4. UV-IR mixing

Having obtained the effective superpotential, in this section, we shall discuss how physics changes in various limits of coupling parameters. The first case is that both Θ and λ are nonzero. In this case, we can integrate the overall modulus τ and simplify the effective superpotential eq. (3.4) further. The second case is that $\Theta = 0$ (so the y coordinates commute). Here we observe the dipole effect in superspace. The third case is that $\Theta = 0$ and $\lambda \to \infty$, where the open Wilson line collapses to a point. We recover the effective superpotential of the Wess-Zumino model deformed by C only.

4.1 Full non(anti)commutativity

Consider first the situation when we have both noncommutativity of Grassmann-even coordinates and nonanticommutativity of Grassmann-odd coordinates. We can then perform integration over the overall modulus τ in eq. (3.3) or, equivalently, in eq. (3.4) after expanding the open Wilson line $\mathcal{W}[\Phi]$ in powers of Φ's and bringing down τ's from the exponent. The integration involved is of the form

$$\int \mathrm{d}\tau \tau^{N-2} \mathcal{K}(\tau, K_0) = 2M^{N-1} \left[\frac{\overline{m}}{4\lambda} \mathbf{K}_{1-N}(2M) + \kappa_0^2 \frac{L}{M} \mathbf{K}_{2-N}(2M) \right], \tag{4.1}$$

where $\mathbf{K}_n(z)$ is the modified Bessel function. A relevant property of the modified Bessel function is that asymptotically at large z,

$$\mathbf{K}_n(z) \sim \sqrt{\frac{\pi}{2z}} e^{-|z|} \left[1 + \mathcal{O}\left(\frac{1}{|z|} \right) \right],$$

which is *independent* of n. So, in the limit we are taking eq. (3.2), integrating explicitly over the overall modulus results in

$$\int \mathrm{d}^6 Y \, W_{\text{eff}} = \left(\frac{m\overline{m}}{4\pi^2} \right) \int \mathrm{d}^6 K_0 W_{K_0}[\Phi]_U \left(\frac{\overline{m}}{4\lambda M} + \frac{L}{M^2} \kappa_0^2 \right) \mathbf{K}_0(2M) W_{-K_0}[\Phi]_T \tag{4.2}$$

where we have defined a new open Wilson line with superspace size determined by M, via the kinematic factor $\sqrt{h} \equiv \frac{q}{m} M$:

$$W_{K_0}[\Phi]_U = \mathcal{P}_t \left[- \int \mathrm{d}^6 Z \exp\left(\int_0^1 \mathrm{d}\sigma \sqrt{h} \Phi(y + \sigma(\Theta k_0), \theta - \sigma(C\kappa_0)) \right) \star e^{-ik_0 \cdot y - \kappa_0 \theta} \right]_U, \tag{4.3}$$

and the twisted function $W_{K_0}[\Phi]_T$ just has an extra minus sign in every phase factor.

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It is instructive to compare the general expression of the effective superpotential eqs. (3.4), (3.5) with the present one, eqs. (4.2), (4.3). In fact, the expression eq. (4.3) is almost identical with the one given in [15] for ordinary noncommutative field theory, where $\sqrt{h} = \frac{qM}{m}$ plays the role of the metric factor in one-dimensional parameter space and equals $|\dot{y}(\sigma)|$. In the latter, $y(\sigma)$ sets the metric factor since it is a translation-invariant interval in noncommutative space, so we should expect the appearance of a supertranslation-invariant interval in the present case. Indeed it is so and is determined by M, as we now explain.

In the ordinary $\mathcal{N} = 1$ superspace $(x, \theta, \overline{\theta})$, the supertranslation invariant interval is defined by

$$(\Delta s)^2 = w^\alpha w_\alpha \quad \text{where} \quad w = (x_2 - x_1) + i\theta_1\sigma(\overline{\theta}_2 - \overline{\theta}_1) - i(\theta_2 - \theta_1)\sigma\overline{\theta}_1 \,.$$

It is invariant under a general supertranslation,

$$x^m \to x^m + i(\theta\sigma\overline{\zeta} - \zeta\sigma\overline{\theta})^m + \varepsilon^m \,, \qquad \theta \to \theta + \zeta \,, \quad \overline{\theta} \to \overline{\theta} + \overline{\zeta} \,.$$

However, once the $\mathcal{N} = 1$ superspace is reduced to $\mathcal{N} = 1/2$ by the non(anti)commutativity, the latter's chiral coordinates $(y = x + i\theta\sigma\overline{\theta}, \theta)$ are the unique choice of local coordinates. This is because the reduced supersymmetry then requires that $\overline{\zeta} = 0$, and $\overline{\theta}$ is no longer a part of $\mathcal{N} = 1/2$ superspace coordinates. It then follows that Δy is the unique interval invariant under supertranslation in $\mathcal{N} = 1/2$ superspace:

$$y^m \to y^m + \varepsilon^m \,, \qquad \theta^\alpha \to \theta^a + \zeta^\alpha \,.$$

Now that we have understood that Δy is the only $\mathcal{N} = 1/2$ supertranslation invariant length, emergence of the characteristic length $\dot{y}(\sigma) = |\Theta k_0|$ in the open Wilson line eq. (4.3) is readily understood. Recall the modulus integral

$$\int \mathrm{d}\tau \tau^{N-2} e^{-\tau - \frac{M^2}{\tau}} \,.$$

Because both τ and $1/\tau$ appear in the exponential, the dominant contribution comes from the saddle point, where

$$\tau = \frac{M^2}{\tau} \Longrightarrow \tau = M \,.$$

4.2 Nonanticommutative limit

We next consider the limit $\Theta^{mn} = 0$. In this limit, only the θ coordinate is nonanticommutative while the y coordinate is commutative. In this case, we should *not* do the overall moduli integration $\int \mathrm{d}\tau$, but keep the form (3.4) (3.5). The reason is that since $M = 0$, there is no $1/\tau$ term in the exponential. Thus there is *no* characteristic length to dominate the integral, and we must sum all contributions over τ. This explains also why τ shows up in the definition of the open Wilson line. The dependence of the open Wilson line on the overall modulus τ is new compared to the familiar understanding from the ordinary noncommutative field theory [4].

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Besides the new form of the open Wilson line, there are also a few new features related to the fermionic non-anticommutativity. Like the dipole effect in the ordinary noncommutative field theory [18], we see here the dipole effect in θ coordinates too. This is quite generic. Consider, for instance, a 'super-dipole' whose dipole moment is proportional to ζ. Denote the dipole's constituents by a wave function $\Phi(\theta)$. Then, the form-factor of the 'super-dipole' is given by Fourier transform:

$$\begin{aligned} I &\equiv \int \mathrm{d}^2\theta \Phi(x,\theta) \star \Phi(x,\theta+\zeta) \star e^{\kappa_0\theta} \\ &= (-4) \int \mathrm{d}^2\kappa_2 \Phi(x,\kappa_0-\kappa_2)\Phi(x,\kappa_2) e^{-\kappa_2\zeta} e^{-\frac{1}{2}\kappa_0\wedge\kappa_2} \,. \end{aligned}$$

If the distribution is $\Phi(x,\theta) = \Phi(x)\delta^2(\theta)$, we have $\Phi(x,\kappa) = \Phi(x)$, so that

$$I = \Phi(x)^2 \delta^2 \left(\zeta + \frac{1}{2}(C\kappa_0)\right).$$

Thus, the dipole moment is not fixed but is proportional to the center-of-mass momentum κ_0:

$$\Delta\theta^\alpha = \zeta^\alpha \sim C^{\alpha\beta}\kappa_{0\beta} \,,$$

exhibiting precisely the dipole relation eq. (1.5).

4.3 Deformed supersymmetry limit

It is also of interest to take the limit $\lambda \to \infty$. This is the limit that the $\mathcal{N} = 1/2$ kinetic supermultiplet is suppressed and that the $\mathcal{N} = 1$ supersymmetry is restored through smooth extrapolation as the non(anti)commutativity $C^{\alpha\beta}$ is taken to zero. For simplicity, we will also take $\Theta^{mn} \to 0$.

In the $\lambda \to \infty$ limit, the concerned part of the cross-channel propagator is

$$\left[\frac{\overline{m}}{4\lambda} + \frac{L}{\tau}\kappa_0^2\right] \tag{4.4}$$

in eq. (3.4). In terms of superspace coordinates, $\kappa_\alpha = Q_a$, so in the propagator, there is one term with an insertion of Q^2, and one without. We know [12] that the non(anti)commutative $\star$-products produce only even powers of Q^2 (because $C^{\alpha\beta}Q_\alpha\Phi Q_\beta\Phi = 0$). With both of these terms present, all possible powers of Q^2 are produced, both odd and even (up to one less than the total number of Φ's). Thus, all terms generated in W_{eff} appear already at one loop.

In the $\lambda \to \infty$ limit, the second term in eq. (4.4) dominates, so the $\Phi Q^2\Phi$ term disappears, and the theory reduces to the classical part of the deformed Wess-Zumino model discussed in [7]. One loop corrections give only terms like $\Phi^k(Q^2\Phi)^{2l+1}$, as observed in [12, 13]. Also, since κ_0^2 appears in the cross-channel propagator explicitly, we can not bring any further factors of κ_0 from the open Wilson line. Thus, one can set the Grassmann coordinate $(\theta - C\kappa_0\,\sigma)$ of the open Wilson line effectively to θ, viz. the open Wilson line is collapsed to a local operator (with a base point at θ). This explains why we do not see the

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open Wilson line if the kinetic multiplet term is absent. If in addition $\Theta = 0$, the effective superpotential reads

$$\int \mathrm{d}^6 Y \, W_{\text{eff}}[\Phi_0] = \frac{(m\overline{m})^2}{4\pi^2} \int_0^\infty \frac{\mathrm{d}\tau}{\tau^3} e^{-\tau} \int \mathrm{d}^6 Y \, \mathcal{W}_\tau[\Phi_0] \left(-\frac{1}{4}Q^2\right) \mathcal{W}_\tau[\Phi_0] \tag{4.5}$$

where

$$\mathcal{W}_\tau[\Phi(Y)] = \mathrm{Tr}\, \exp_\star \left[-\tau \frac{g}{m} \Phi_0(Y)\right] .$$

is the open Wilson line, now collapsed to a point.

As it stands, the overall-modulus integral in eq. (4.5) is divergent at $\tau \sim 0$ for terms up to quadratic in Φ_0. This can be understood from the observation that, as Θ^{mn} is now set to zero, nonplanar diagram of Grassmann-odd $\star$-product no longer exhibits the UV-IR mixing for Grassmann-even momenta. As the divergence would be absent in case Θ^{mn} is nonzero, a consistent treatment would be to treat the logarithmically divergent operator $\Phi Q^2 \Phi$ as a state created by the open Wilson line on the mass-shell. In other words, with a change of variables $s = 1/\tau$, the logarithmic divergence originates from $\int \mathrm{d}s \exp(-P^2 s)$ at $P^2 \sim 0, \ s \sim \infty$.

5. Planar contribution

So far, we have focused mainly on the nonplanar contribution to the effective superpotential W_{eff}. For example, the N-point function involves n untwisted U and $(N-n)$ twisted T interactions, attached in double-line notation to inner and outer index loops, respectively. The planar contributions originate from two exceptional cases, $n = 0$ and $n = N$. In the double-line notation, they correspond to either all U or all T interactions, attached only one of the two index loops. Equivalently, they correspond to setting either of the two open Wilson lines in the effective superpotential eq. (3.4) to unity. Since the corresponding index loop is free, these planar amplitudes would be proportional to $\mathrm{Tr}\, 1 = N_c$.

5.1 Planar combinatorics

Consider again the combinatorics. Because the free index-loop amounts to setting the corresponding open Wilson line to unity, we find for the net-momentum flow $k_0 = 0$ and $\kappa_0 = 0$. This vanishing has two effects. First, the path-ordered $\star$-product reduces to the standard $\star$-product eq. (1.3). Second, the argument of the modified Bessel function in the cross-channel propagator vanishes. More precisely, if we introduce an ultraviolet cutoff Λ, then the argument needs to be set to Λ^{-2}, and in the end we take the limit $\Lambda \to \infty$. First we recall the small argument behavior of the modified Bessel function:

$$\mathbf{K}_n(2M) \sim \frac{1}{2}\Gamma(|n|) M^{-|n|} \tag{5.1}$$

for $n \neq 0$, and

$$\mathbf{K}_0(2M) \sim -\log M \tag{5.2}$$

for $n = 0$. One again finds ultraviolet divergence for lower-point amplitudes, so we implicitly perform a suitable renormalization.

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The small argument behavior eqs. (5.1), (5.2) concerning the cross-channel propagator can be understood directly from the overall modulus integral. Expansion of the open Wilson line contributes to the N-point function a factor of $\tau^N/N!$, so the overall-modulus integral yields

$$\int \frac{\mathrm{d}\tau}{\tau^2}\,\frac{\tau^N}{N!} \sim \frac{(N-2)!}{N!}\,.$$

We then decompose this factor into partial fractions,

$$\frac{(N-2)!}{N!} = \frac{1}{N(N-1)} = -\left(\frac{1}{N} - \frac{1}{(N-1)}\right),$$

and resum each term separately over N. Thus, the planar contribution to the effective superpotential is now given by

$$W_{\text{eff}}[\Phi_0]\Big|_{\text{planar}} \simeq -N_c\,(m+g\Phi_0)\star\log_\star(m+g\Phi_0)\,. \tag{5.3}$$

The functional form of the planar contribution eq. (5.3) is interesting. Suppose we had started with, instead of the action eq. (2.1), a new action in which Q_α is replaced by $Q^2 + W^\alpha W_\alpha$, where W_α refers to a spinorial background field. The background field may be viewed as a gauge field associated with super-translation. We then observe diagrammatically that the net effect is to replace $g\Phi_0 \to g\Phi_0 + \mathcal{W}^\alpha\mathcal{W}_\alpha$. We can then even turn off the background field Φ_0, set $m=0$, and obtain the planar contribution to the effective superpotential of $\mathcal{W}^\alpha\mathcal{W}_\alpha$ as

$$W_{\text{eff}}[\mathcal{W}^\alpha\mathcal{W}_\alpha]\Big|_{\text{planar}} \sim -N_c\,\mathcal{W}^a\mathcal{W}_\alpha\log\left(\mathcal{W}^\alpha\mathcal{W}_\alpha\right).$$

Intriguingly, this is precisely the same functional form as the Veneziano-Yankielowicz glueball superpotential [20], if we identify W_α as the gauge superfield. This may be an indication that even the non-analytic part of the nonperturbative glueball superpotential may be derived from perturbative dynamics in the context of the Dijkgraaf-Vafa correspondence. This is not inconceivable, since the logarithmic term in eq. (5.3) originates essentially from dimensional transmutation, viz. the Coleman-Weinberg mechanism, and this is what also underlies the original derivation of the glueball superpotential [20].

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A. Notations and conventions

A.1 Fourier transform in superspace

Here, we collect our notation and present some useful formulas. The $\mathcal{N} = 1/2$ superspace coordinates and momenta are abbreviated as Y^a and K_a, respectively

$$Y^a \equiv (y^m, \theta^\alpha) \quad \text{and} \quad K_a \equiv (ik_m, \kappa_\alpha)\,.$$

We adopt the following convention of the superspace Fourier transformation:

$$\widetilde{\Phi}(K) = \int \mathrm{d}^4y\ \mathrm{d}^2\theta\ e^{-iky-\kappa\theta}\ \Phi(Y) \equiv \int \mathrm{d}^6Y e^{-KY}\ \Phi(Y)$$
$$\Phi(Y) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{d}^2\kappa}{(i/2)^2} e^{iky+\kappa\theta}\widetilde{\Phi}(K) \equiv \int \mathrm{d}^6K\ e^{KY}\ \widetilde{\Phi}(K)\,.$$

In the convention adopted, we abbreviate the Dirac δ-functions as

$$\int \mathrm{d}^6K\ e^{KY} = \delta^4(y)\delta^2(\theta) \equiv \delta^6(Y)$$
$$\int \mathrm{d}^6Y\ e^{-KY} = (2\pi)^4\delta^4(k)(i/2)^2\delta^2(\kappa) \equiv \delta^6(K)\,.$$

A.2 The lagrangian in momentum space

The lagrangian is given by

$$S = \int \mathrm{d}^6Y \frac{1}{2}\Phi(Y)\left[\frac{1}{4\lambda}Q^2 - \frac{1}{\overline{m}}\Box + m\right]\Phi(Y) + \frac{g}{3}\Phi(Y)\star\Phi(Y)\star\Phi(Y)\,.$$

The propagator is given by

$$\langle\Phi(Y)\Phi(Y')\rangle = \left[-\frac{1}{\overline{m}}\Box_y + \frac{1}{4\lambda}Q^2 + m\right]^{-1}\delta^6(Y-Y')$$

in configuration superspace, or equivalently in momentum space,

$$\langle\widetilde{\Phi}(K)\widetilde{\Phi}(K')\rangle = \left[\frac{1}{\overline{m}}k^2 + \frac{1}{4\lambda}\kappa^2 + m\right]^{-1}\delta^6(K-K')\,.$$

Using these results, the lagrangian in momentum superspace is obtained as

$$S = \frac{1}{2}\int \mathrm{d}^6K_1\mathrm{d}^6K_2\delta^6(K_1+K_2)\left[\frac{1}{\overline{m}}k_1^2 + \frac{1}{4\lambda}\kappa_1^2 + m\right]\widetilde{\Phi}(K_1)\widetilde{\Phi}(K_2) +$$
$$+\frac{g}{3}\int\prod_{i=1,2,3}\mathrm{d}^6K_i\widetilde{\Phi}(K_i)\ \delta^6(K_1+K_2+K_3)e^{-\frac{1}{2}\sum_{i<j}\kappa_i\wedge\kappa_j - \frac{i}{2}\sum_{i<j}k_i\wedge k_j}\,,$$

where we have defined

$$\kappa_1\wedge\kappa_2 = \kappa_{1\alpha}C^{\alpha\beta}\kappa_{2\beta} = -\kappa_2\wedge\kappa_1\,, \qquad (C\kappa)^\alpha = C^{\alpha\beta}\kappa_\beta\,,$$
$$p_1\wedge p_2 = p_{1m}\Theta^{mn}p_{2n} = -p_2\wedge p_1\,, \qquad (\Theta p)^m = \Theta^{mn}p_n\,.$$

In fact we can define

$$K_1\wedge K_2 \equiv (ik_1)_m\Theta^{mn}(ik_2)_n + \kappa_{1\alpha}iC^{\alpha\beta}\kappa_{2\beta}\,,$$

and then the phase factor is just $\sum_{i<j}\frac{i}{2}K_i\wedge K_j$. However, since we may deal with the bosonic and fermionic parts separately, we will use the component form or compact form as convenient.

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B. Evaluation of the momentum integrals

Here we present some details of the derivation of eq. (3.3).

We have defined the net-flow momenta across the channel

$$k_0 \equiv \sum_{i\in\{U\}} k_i = -\sum_{i\in\{T\}} k_i\,, \qquad \kappa_0 \equiv \sum_{i\in\{U\}} = -\sum_{i\in\{T\}} \kappa_i\,,$$

where $i \in \{U\}/\{T\}$ means that vertex i is un/twisted. Then,

$$\sum_{i=1}^{N} \frac{\epsilon_i}{2} k_i = k_0\,, \qquad \sum_{i=1}^{N} \frac{\epsilon_i}{2} \kappa_i = \kappa_0\,.$$

Using the antisymmetry of the wedge product and the overall momentum conservation, one can see that

$$\sum_{i=1}^{N} \frac{i\epsilon_i}{2} k_i \wedge \left(p + \sum_{j=1}^{i} k_j \right) = ik_0 \wedge p + \frac{i}{2} \sum_{j\le i} \frac{\epsilon_i + \epsilon_j}{2} k_i \wedge k_j$$

and similarly

$$\sum_{i=1}^{N} \frac{\epsilon_i}{2} \kappa_i \wedge \left(\pi + \sum_{j=1}^{i} \kappa_j \right) = \kappa_0 \wedge \pi + \frac{1}{2} \sum_{j\le i} \frac{\epsilon_i + \epsilon_j}{2} \kappa_i \wedge \kappa_j\,.$$

Move the propagator into the exponent by introducing Schwinger parameters s_j:

$$\begin{aligned} G_N \;=\; & -\frac{1}{N!}(-g\overline{m})^N \int \prod_{i=1}^{N} \mathrm{d}^4 k_i \mathrm{d}^2 \kappa_i \widetilde{\Phi}_0(k_i, \kappa_i) \delta^4(k_1 + \cdots + k_N) \delta^2(\kappa_1 + \cdots + \kappa_N) \times \\ & \times \int \mathrm{d}^4 p \mathrm{d}^2 \pi \int_0^\infty \mathrm{d}s_1 \cdots \mathrm{d}s_N \prod_{i=1}^{N} \exp\left[-s_i \left(\left(p + \sum_{j\le i} k_j \right)^2 + \frac{\overline{m}}{4\lambda} \left(\pi + \sum_{j\le i} \kappa_j \right)^2 + m\overline{m} \right) \right] \times \\ & \times \prod_{i=1}^{N} \exp\left[ik_0 \wedge p + \frac{i}{2} \sum_{j\le i} \frac{\epsilon_i + \epsilon_j}{2} k_i \wedge k_j + \kappa_0 \wedge \pi + \frac{1}{2} \sum_{j\le i} \frac{\epsilon_i + \epsilon_j}{2} \kappa_i \wedge \kappa_j \right]. \end{aligned}$$

To continue, we group the phase factors into bosonic and fermionic parts.

(a) The phase factor of the bosonic part is given by

$$-p^2 \left(\sum_i s_i \right) - 2p \sum_i s_i \sum_j k_j - \sum_i s_i \left(\sum_j k_j \right)^2 + ik_0 \wedge p + \frac{i}{2} \sum_{j\le i} \frac{\epsilon_i + \epsilon_j}{2} k_i \wedge k_j\,.$$

Complete the square and perform the loop momentum p-integral. This gives an overall factor of $\Omega_3/2(\sum_i s_i)^2$ where Ω_3 is the volume of unit 3-sphere. Redefine the moduli parameters as follows:

$$s \equiv \sum_{i=1}^{N} s_i\,; \qquad \sigma_i \equiv \frac{1}{s} \sum_{i=i}^{N} s_i\,.$$

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In these new variables, we have

$$\sum_i s_i \sum_j k_j = s \sum_i \sigma_i k_i$$
$$\sum_i s_i \left(\sum_j k_j \right)^2 = s \sum_{i=1}^{N} \sigma_i k_i \left(-k_i + 2 \sum_{j=1}^{i} k_j \right)$$
$$\int_0^\infty \mathrm{d}s_1 \cdots \mathrm{d}s_N = \int_0^\infty \mathrm{d}s \; s^{N-1} \prod_{i=2}^{N} \int_0^{\sigma_{i-1}} \mathrm{d}\sigma_i \, .$$

The remaining bosonic phase can be rewritten as

$$s \left(\sum_i \sigma_i k_i \right)^2 - s \sum_i \sigma_i k_i \left(-k_i + \sum_{j \le i} k_j \right) + i(\Theta k_0) \sum_i \sigma_i k_i - \frac{1}{4s} (\Theta k_0)^2 + \\ + \frac{i}{2} \sum_{j \le i} \frac{\epsilon_i + \epsilon_j}{2} k_i \wedge k_j \, .$$

(b) The phase factor of the fermionic part is given by

$$-\frac{\overline{m}}{4\lambda} \left(\sum s_i \right) \left(\pi + \sum_j \kappa_j \right)^2 + \kappa_0 \wedge \pi + \frac{1}{2} \sum_{j \le i} \frac{\epsilon_i + \epsilon_j}{2} \kappa_i \wedge \kappa_j \, .$$

The result of the π integral, in terms of the new moduli, is an overall factor of $-(\overline{m}/4\lambda)s$ and a phase of

$$s \frac{\overline{m}}{4\lambda} \left(\sum_i \sigma_i \kappa_i \right)^2 + 2s \frac{\overline{m}}{4\lambda} \sum_i \sigma_i \kappa_i \sum_j \kappa_j - (C\kappa_0) \sum_i \sigma_i \kappa_i + \\ + \frac{4\lambda}{4\overline{m}} \frac{(C\kappa_0)^2}{s} + \frac{1}{2} \sum_{j \le i} \frac{\epsilon_i + \epsilon_j}{2} \kappa_i \wedge \kappa_j \, .$$

Now we take the limit of large non(anti)commutativity and small external momenta:

$$k_i, \kappa_i \to \mathcal{O}(\epsilon) \quad \text{and} \quad \Theta^{mn}, C^{\alpha\beta} \to \mathcal{O}(\epsilon^{-2}) \, .$$

Evidently only the last three terms in each case remain. Equation eq. (3.3) thus follows immediately.

References

[1] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[2] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

JHEP08(2003)001

[3] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *J. High Energy Phys.* **02** (2000) 020 [hep-th/9912072];
M. Van Raamsdonk and N. Seiberg, *Comments on noncommutative perturbative dynamics*, *J. High Energy Phys.* **03** (2000) 035 [hep-th/0002186].

[4] S.-J. Rey and R. von Unge, *S-duality, noncritical open string and noncommutative gauge theory*, *Phys. Lett.* **B 499** (2001) 215 [hep-th/0007089];
S.R. Das and S.-J. Rey, *Open wilson lines in noncommutative gauge theory and tomography of holographic dual supergravity*, *Nucl. Phys.* **B 590** (2000) 453 [hep-th/0008042];
D.J. Gross, A. Hashimoto and N. Itzhaki, *Observables of non-commutative gauge theories*, *Adv. Theor. Math. Phys.* **4** (2000) 893 [hep-th/0008075];
N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, *Wilson loops in noncommutative Yang-Mills*, *Nucl. Phys.* **B 573** (2000) 573 [hep-th/9910004].

[5] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[6] H. Ooguri and C. Vafa, *The c-deformation of gluino and non-planar diagrams*, hep-th/0302109; *Gravity induced c-deformation*, hep-th/0303063.

[7] N. Seiberg, *Noncommutative superspace, $N = 1/2$ supersymmetry, field theory and string theory*, *J. High Energy Phys.* **06** (2003) 010 [hep-th/0305248].

[8] N. Berkovits and N. Seiberg, *Superstrings in graviphoton background and $N = 1/2 + 3/2$ supersymmetry*, *J. High Energy Phys.* **07** (2003) 010 [hep-th/0306226].

[9] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, hep-th/0302078.

[10] D. Klemm, S. Penati and L. Tamassia, *Non(anti)commutative superspace*, *Class. and Quant. Grav.* **20** (2003) 2905 [hep-th/0104190].

[11] H. Kawai, T. Kuroki and T. Morita, *Dijkgraaf-Vafa theory as large-N reduction*, *Nucl. Phys.* **B 664** (2003) 185 [hep-th/0303210].

[12] R. Britto, B. Feng and S.-J. Rey, *Deformed superspace, $N = 1/2$ supersymmetry and (non)renormalization theorems*, *J. High Energy Phys.* **07** (2003) 067 [hep-th/0306215].

[13] S. Terashima and J.-T. Yee, *Comments on noncommutative superspace*, hep-th/0306237.

[14] M.R. Garousi, *Non-commutative world-volume interactions on D-branes and Dirac-Born-Infeld action*, *Nucl. Phys.* **B 579** (2000) 209 [hep-th/9909214]; *Supergravity couplings of noncommutative D-branes*, *Nucl. Phys.* **B 615** (2001) 169 [hep-th/0101016].

[15] Y. Kiem, S.-J. Rey, H.-T. Sato and J.-T. Yee, *Open Wilson lines and generalized star product in nocommutative scalar field theories*, *Phys. Rev.* **D 65** (2002) 026002 [hep-th/0106121]; *Anatomy of one-loop effective action in noncommutative scalar field theories*, *Eur. Phys. J.* **C 22** (2002) 757 [hep-th/0107106];
Y. Kiem, S. Lee, S.-J. Rey and H.-T. Sato, *Interacting open Wilson lines in noncommutative field theories*, *Phys. Rev.* **D 65** (2002) 046003 [hep-th/0110215].

[16] S.-J. Rey, *Exact answers to approximate questions: noncommutative dipoles, open Wilson lines and UV-IR duality*, hep-th/0207108.

[17] R. Dijkgraaf, M.T. Grisaru, C.S. Lam, C. Vafa and D. Zanon, *Perturbative computation of glueball superpotentials*, hep-th/0211017.

JHEP08(2003)001

[18] D. Bigatti and L. Susskind, *Magnetic fields, branes and noncommutative geometry*, *Phys. Rev.* **D 62** (2000) 066004 [hep-th/9908056].

[19] Y. Kiem, S. Lee, S.-J. Rey and H.-T. Sato, *Interacting open Wilson lines in noncommutative field theories*, *Phys. Rev.* **D 65** (2002) 046003 [hep-th/0110215].

[20] G. Veneziano and S. Yankielowicz, *An effective lagrangian for the pure $N = 1$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 113** (1982) 231.

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